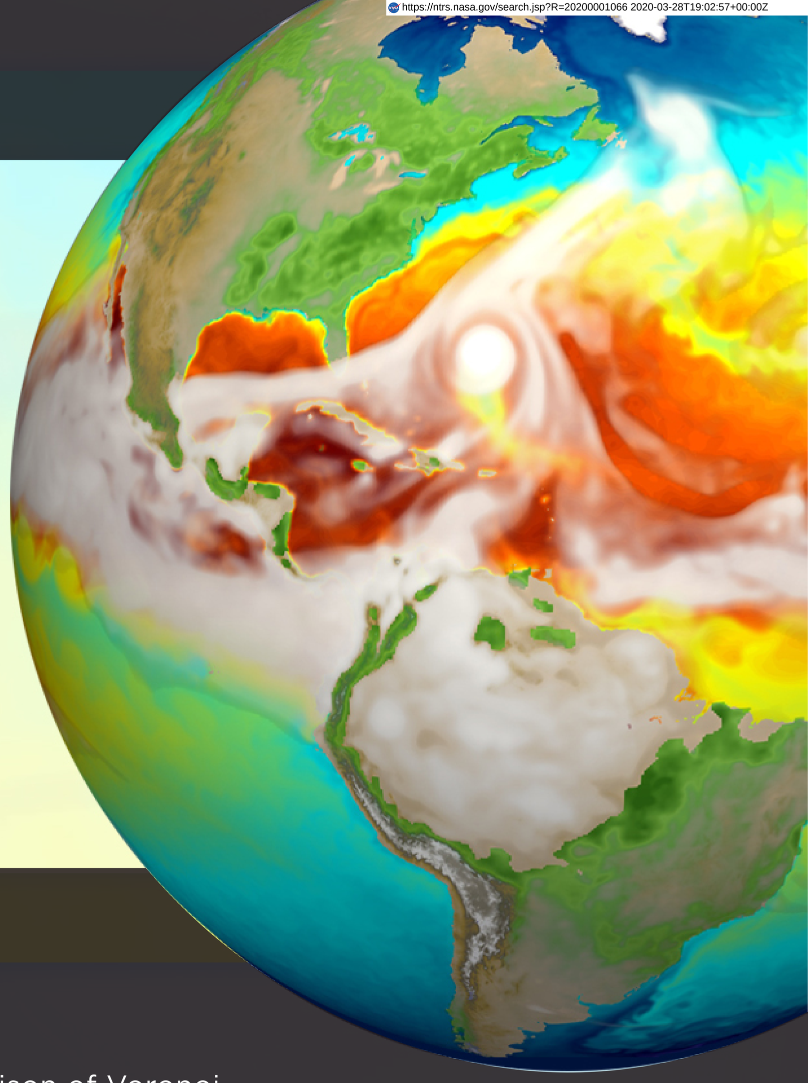

USING POWER DIAGRAMS TO BUILD OPTIMAL UNSTRUCTURED MESHES FOR C-GRID MODELS

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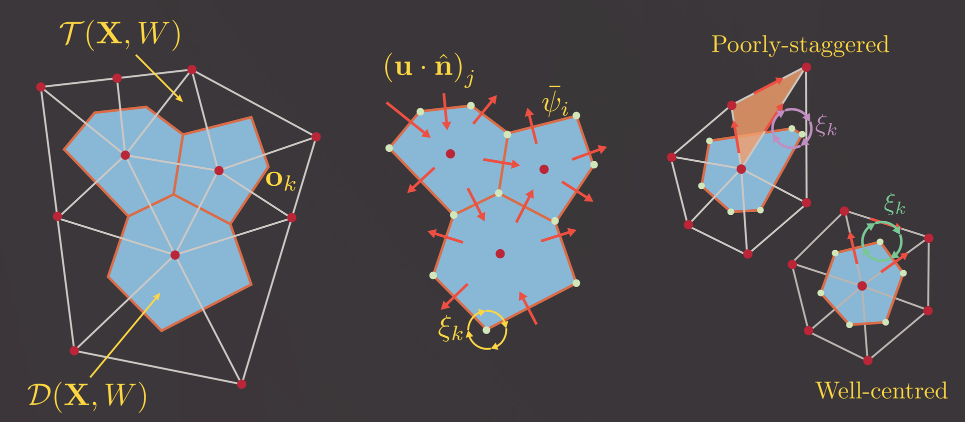
STAGGERED FINITE-VOLUME SCHEMES

The Model for Prediction Across Scales (MPAS)[8,9] for Ocean (-O), Sea-Ice (-SI) and Land-Ice (-LI), in addition to the Coastal Ocean Marine Prediction Across Scales (COMPAS) are two novel general circulation models designed to resolve coupled ocean-ice dynamics over variable spatial scales using non-uniform unstructured grids. Both models are based on a conservative mimetic finite-difference/volume formulation (TRiSK)[8], in which staggered momentum, vorticity and mass-based degrees- of-freedom are distributed over an orthogonal 'primal-dual' mesh.

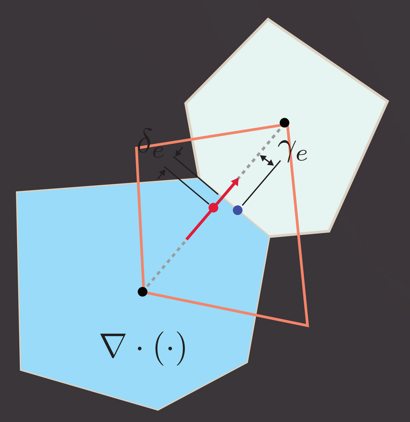


Anatomy of TRiSK: (a) an orthogonal primal-dual grid, (b)finite-difference /volume scheme of Ringler et al, and (c) comparison of *well-centred* and *poorly-staggered* configurations. In (c), configurations differ in the relationship between dual vertices and primal triangles. Reconstruction of vorticity breaks down in poorly- staggered cases.

The accuracy and performance of TRiSK-based models is a strong function of the *quality* of the underlying unstructured grid on which the simulation is run; placing significant pressure on the associated grid generation workflow.

MESH VS DISCREDISATION ERROR

The accuracy of the $\nabla\cdot(\cdot)$, $\nabla(\cdot)$, $\nabla\times(\cdot)$ operators is a function of the geometry of the mesh staggering:

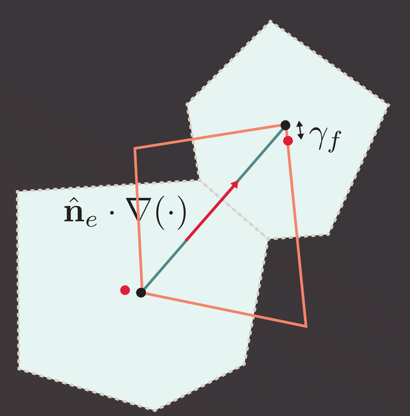


Leading errors in $\nabla\cdot(\mathbf{u}\,\psi)$:

$$\int_{d_i} \nabla\cdot(\mathbf{u}\,\psi)\ \mathrm{d}A = \oint_{\partial d_i} (\mathbf{u}\cdot\hat{\mathbf{n}})\,\psi\ \mathrm{d}s$$

$$\simeq \sum_{e=1}^{n} \int_e (\mathbf{u}\cdot\hat{\mathbf{n}})_e\,\psi_e\ \mathrm{d}l$$

Only 2nd order accurate if $\delta_e = 0,\ \gamma_e = 0$.



Leading errors in $\nabla(\Phi)$:

normal component: $\hat{\mathbf{n}}_e\cdot\nabla(\cdot)$

$$\hat{\mathbf{n}}_e\cdot\nabla\Phi \simeq l_e^{-1}\,(\Phi_2-\Phi_1)$$

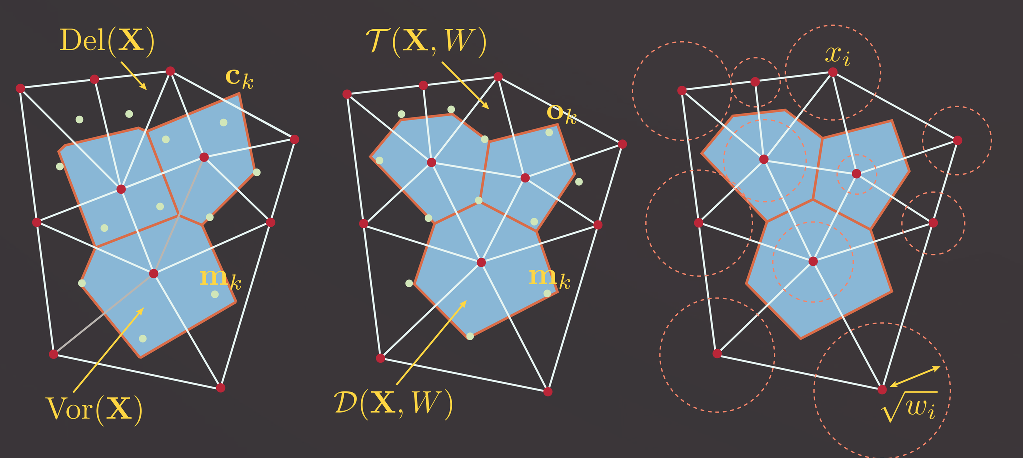
Only 2nd-order accurate if $\gamma_f = 0$.

(otherwise interpolaiton is not centred!)

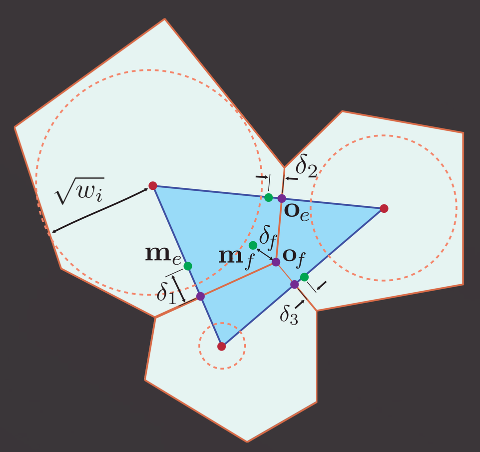
Optimal meshes should minimise various 1st-order error terms: triangle/cell edge offsets, vertex/centroid offsets, and centre/centroid offsets.

OPTIMAL PRIMAL-DUAL MESH GENERATION

Building on the conventional Centroidal Voronoi Tessellation (CVT) framework[5], we have developed a new class of optimal orthogonal grids based on weighted 'Power' diagrams and their associated dual 'Regular' triangulations[1,6]. The presence of an additional set of vertex 'weights' in the Regular-Power formulation provides new opportunities for mesh optimisation — facilitating the construction of 'optimal' staggered grids that exhibit improved characteristics compared to conventional Delaunay-Voronoi configurations. By choosing the weights to regularise the staggering between the polygonal cells and triangles in the mesh, the overall discretisation error of TRiSK can be minimised.



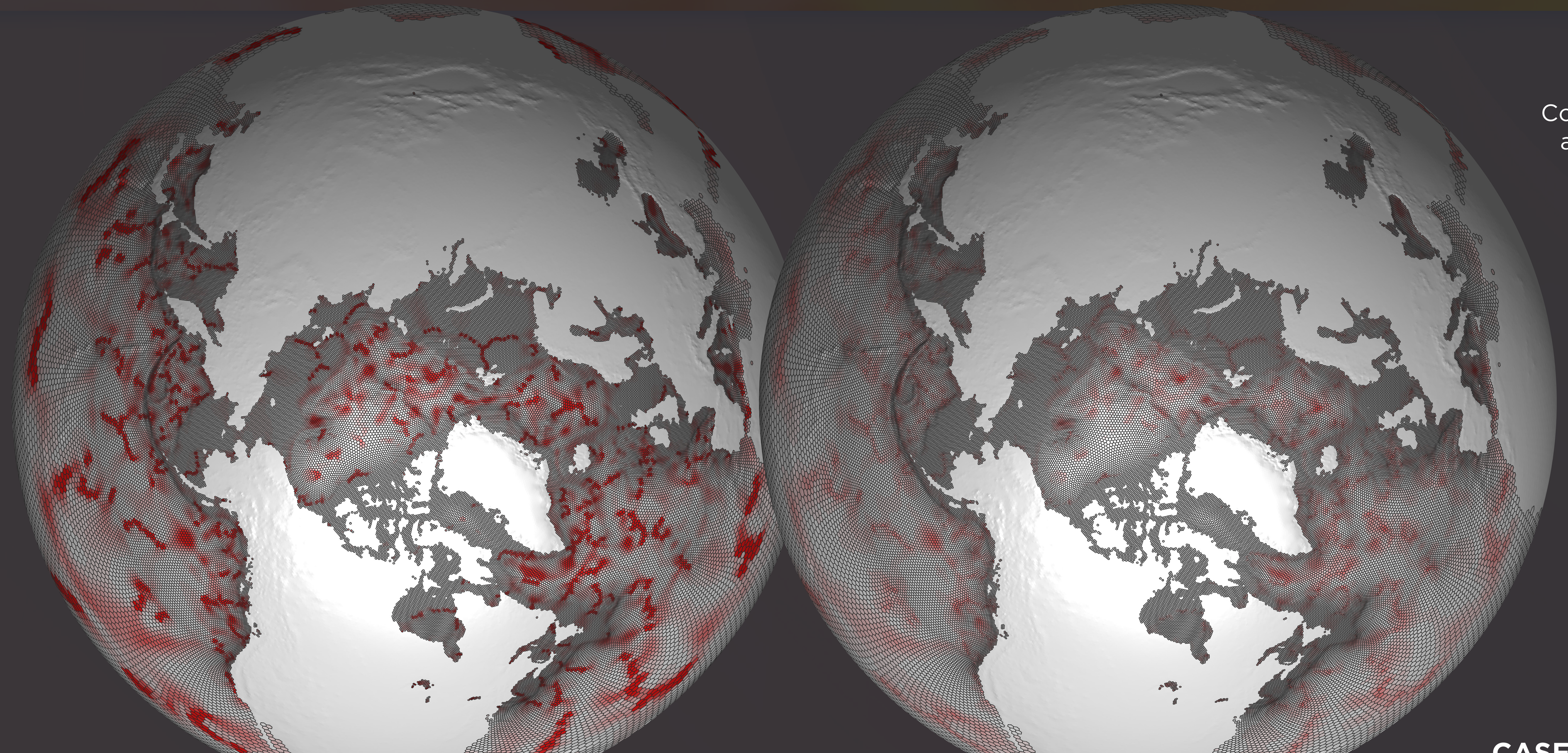
Comparison of conventional and 'optimal' primal-dual grids, showing: (a) a standard Delaunay-Voronoi tessellation, (b) an optimised Regular-Power structure, and (c) the distribution of vertex 'weights' employed in (b). The of the primal-dual tessellation is enhanced in (b) by use of the new methods.



$$\mathcal{Q}_i^{\mathcal{D}}(\mathbf{X},W) = \beta_f \underbrace{\left(1-\left(\frac{\delta_f}{\bar{l}_f}\right)^2\right)}_{\text{'defect' at face}} + \beta_e \underbrace{\left(\frac{1}{3}\sum_{e=1}^{3} 1-\left(\frac{\delta_e}{l_e}\right)^2\right)}_{\text{mean 'defect' at edges}},$$

$$\text{with}\quad \delta_f = \|\mathbf{o}_f-\mathbf{m}_f\|\,,\quad \delta_{1,2,3} = \|\mathbf{o}_{1,2,3}-\mathbf{m}_{1,2,3}\|\,.$$

find $\mathbf{X}, W,$ such that $\min \mathcal{Q}_i^{\mathcal{D}}(\mathbf{X},W)\ \forall\ \tau_i \in \mathcal{T}(\mathbf{X},W)$ is maximised.



Comparison of Voronoi- and Power-based meshes, showing normalised measures of mesh staggering error. The optimisation of weights in the Power mesh can be seen to reduce the magnitude of the staggering error.

JIGSAW MESH GENERATOR

JIGSAW[2] is a new unstructured meshing library supporting the generation of very high-quality, *orthogonal* tessellations for complex geoscientific domains. Based on a combination of 'Frontal' Delaunay-refinement schemes[2,3,4] and hybrid mesh optimisation techniques[1,2], JIGSAW can rapidly generate very high-quality meshes for various global, regional and locally-refined configurations, with support for the MPAS-O/-SI/-LI and COMPAS Earth System Models.

REFERENCES

[1] D. Engwirda: Generalised primal-dual grids for unstructured co-volume schemes, J. Comput. Phys., 375, pp. 155-176, https://doi.org/10.1016/j.jcp.2018.07.025, 2018.

[2] D. Engwirda: JIGSAW-GEO (1.0): locally orthogonal staggered unstructured grid generation for general circulation modelling on the sphere, Geosci. Model Dev., 10, pp. 2117-2140, https://doi.org/10.5194/gmd-10-2117-2017, 2017.

[3] D. Engwirda, D. Ivers, Off-centre Steiner points for Delaunay-refinement on curved surfaces, Computer-Aided Design, 72, pp. 157-171, http://dx.doi.org/10.1016/j.cad.2015.10.007, 2016.

[4] D. Engwirda, Locally-optimal Delaunay-refinement and optimisation-based mesh generation, Ph.D. Thesis, School of Mathematics and Statistics, The University of Sydney, http://hdl.handle.net/2123/13148, 2014.

[5] Q. Du, V. Faber, M. Gunzburger, Centroidal Voronoi tessellations: applications and algorithms, SIAM Rev. 41 (4) pp. 637–676, 1999.

[6] P. Mullen, P. Memari, F. de Goes, M. Desbrun, HOT: Hodge-optimized triangulations, ACM Trans. Graph. 30 (4) pp. 103, 2011.

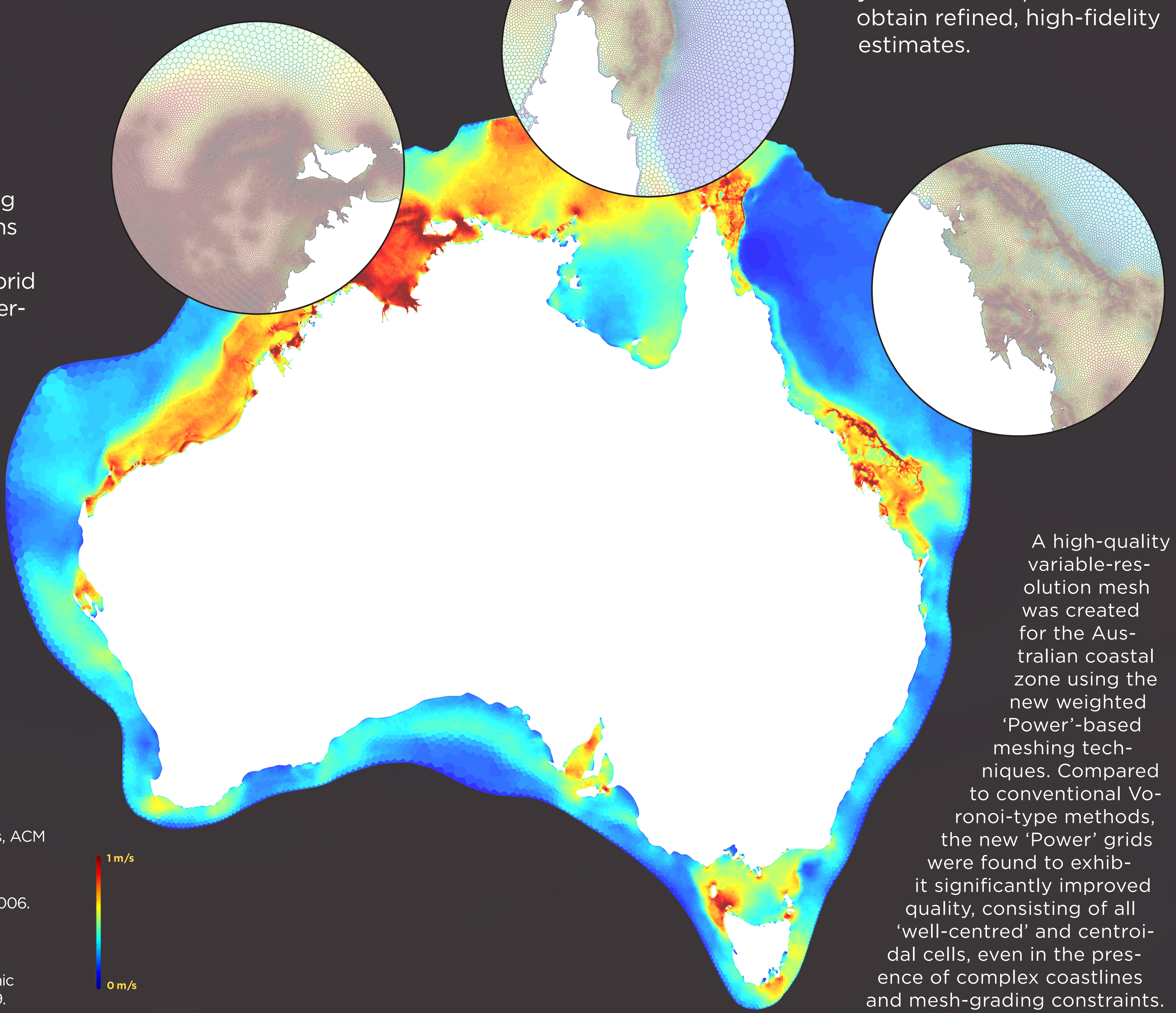
[7] P. O. Persson, Mesh size functions for implicit geometries and PDE-based gradient limiting, Eng. Comput. 22 (2) pp. 95-109, https://doi.org/10.1007/s00366-006-0014-1, 2006.

[8] T. Ringler, M. Petersen, R. Higdon, D. Jacobsen, P. Jones, M. Maltrud, A multi-resolution approach to global ocean modeling, Ocean Model. 69, pp. 211-232, 2013.

[9] J. Thuburn, T. Ringler, W. Skamarock, J. Klemp, Numerical representation of geostrophic modes on arbitrarily structured C-grids, J. Comput. Phys. 228 (22) pp. 8321-8335, 2009.

CASE STUDY: ESTIMATING AUSTRALIA'S TIDAL RESOURCES

We aim to map Australia's tidal energy resources through a characterisation of various tidal amplitude and mean-flow derived metrics. We have employed a multi-scale, unstructured model to balance simulation skill and computational expense — concentrating patches of very high model resolution (750m) in regions of peak tidal capacity, while transitioning to coarse representations (50km) in weakly energised areas. We have employed a 'solution-adaptive' approach: (1) running the model on a relatively coarse and simplistic mesh to obtain initial flows, (2) building a complex, multi-resolution grid adapted to contours of depth-averaged kinetic energy, and (3) running the model a second time using the highly-resolved adapted mesh to obtain refined, high-fidelity estimates.



A high-quality variable-resolution mesh was created for the Australian coastal zone using the new weighted 'Power'-based meshing techniques. Compared to conventional Voronoi-type methods, the new 'Power' grids were found to exhibit significantly improved quality, consisting of all 'well-centred' and centroidal cells, even in the presence of complex coastlines and mesh-grading constraints.